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Lesson $n^{\circ}205$: Complete metric spaces. Examples.

Devs :

- Théorème de Riesz-Fischer
- Théorème de Cauchy-Lipschitz

Références :

1. Gourdon, Analyse
2. James Munkres, Topology
3. Rudin, Analyse réelle et complexe
4. Hirsch Lacombe, Elements d'analyse fonctionnelle
5. Objectif Agrégation
6. Rouvière, Petit guide de calcul différentiel
7. Autre

In all this paper, (X, m) designates a metric space.

1 Definitions and first properties.

1.1 Basics.

Definition 1. *A sequence $(x_n)_{n\in\mathbb{N}}$ of $X^{\mathbb{N}}$ is called a Cauchy sequence if :*

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m, n \geq N \quad d(x_m, x_n) \leq \varepsilon.$$

Example 2.

- A sequence which converges to an element of X it's a Cauchy sequence.
- A Cauchy sequence is always bounded.

Proposition 3. *Let m and m' two equivalent metrics on E. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, m) if and only if it's a Cauchy sequence on (X, m').*

Definition 4. *We say that the metric space X is complete for the metric m if every Cauchy sequence converges to an element of X.*

Example 5.

- The metric space $\mathbb{R}^n$ with the usual euclidian metric is complete.
- The space $\mathbb{Q}$ is not complete, because the Heron sequence defined by $u_0 = 2$ and $u_{n+1} = \frac{1}{2}\left(u_n + \frac{1}{2}\right)$ converges to $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

Remark 6. The notion of completeness is not a topological notion.

For example, the metrics m defined on $\mathbb{R}$ by $\overline{m}(x, y) = |\text{Arctan}(x) - \text{Arctan}(y)|$ defines the same topology as the usual metric, but $(n)_{n\in\mathbb{N}}$ is a Cauchy sequence for $\overline{m}$ (obviously not for the usual metric).

1.2 Properties

Proposition 7. *Let $(X_1, m_1), \ldots, (X_n, m_n)$ be metric spaces. Then the product metric space $X_1 \times \cdots \times X_n$ is complete (for the product metric) if and only if (E_i, m_i) is complete for all i.*

Proposition 8.

- *A complete subspace from a metric space is closed.*
- *A closed subspace from a complete metric space is complete.*
- *A compact metric space is complete.*

Theorem 9. *(Nested intervals theorem)*

Let (E, m) be a complete metric space, and $(F_n)_{n\in\mathbb{N}}$ a decreasing sequence of closed unempty subsets from E, such as $\lim_{n\to+\infty} \delta(F_n) = 0$, where δ designated the diameter of the space F_n.

Then there exist $x \in E$ such as $\bigcap_{n\in\mathbb{N}} F_n = \{x\}$.

1.3 Notion of completion of a metric space.

Definition 10. *If (Y, m') is a metric space, one can define another metric space on the set $\mathcal{B}(X, Y)$ of bounded functions from X to Y, using the metric ρ defined by :*

$$\rho(f, g) := \sup_{x\in X} \{m(f(x)), m(g(x))\}.$$

Theorem 11. *The space $\mathcal{B}(X, Y)$ is a complete metric space.*

Theorem 12. *Let (E, d) be a metric space. There is an isometric imbedding of E into the complete metric space $(\mathcal{B}(E, \mathbb{R}), \rho)$.*

Definition 13. *Let (E,d) be a metric space. If $h: E \to F$ is an isometric imbedding of E into a complete metric space F, then the subspace $\overline{h(X)}$ of Y is a complete metric space. It is called the completion of E.*

Proposition 14. *The completion of a metric space is uniquely determined up to an isometry.*

2 Banach spaces.

In this section, $(E,\|.\|)$ and $(F,\|.\|)$ are normed vector spaces.

Definition 15. *The normed vector space $(E,\|.\|)$ is called a Banach space if it's a complete space for the metric induced by the norm $\|.\|$.*

Theorem 16. *The space $(E,\|.\|)$ is a Banach space if and only if every absolutely convergent series of elements of E is convergent.*

2.1 Fonctionnal spaces.

Theorem 17. *The spaces of $\mathcal{B}(E,F)$ and $\mathcal{C}^0(E,F)$, respectively the space of bounded functions from $(E,\|.\|)$ to $(F,\|.\|)$ and the space of continuous fonctions from $(E,\|.\|)$ to $(F,\|.\|)$, are Banach spaces for the uniform norm.*

Theorem 18. *If K is a compact space, then the space $\mathcal{F}(K,E)$ of functions from K to E is a Banach space.*

Theorem 19. *The space $\mathcal{C}^k([0,1],\mathbb{R})$ with the norm $\|.\|_k$ given by $\|f\|_k := \sum_{i=0}^{k} \|f^{(i)}\|_\infty$ is a Banach space.*

2.2 Continuous linear mappings.

Definition 20. *The set of continuous linear mappings from $(E,\|.\|)$ to $(F,\|.\|)$ is noted $\mathcal{L}(E,F)$. It's a normed vector space for the operator norm $|||.|||$ defined by $|||\varphi||| := \sup_{\|x\|=1} \|f(x)\|$.*

Proposition 21. *The operator norm is sub-multiplicative : $|||\varphi\circ\psi||| \leq |||\varphi|||\cdot|||\psi|||$ for every $\varphi,\psi\in\mathcal{L}(E,F)$.*

Theorem 22. *If $(F,\|.\|)$ is a Banach space, then so is $\mathcal{L}(E,F)$.*

Proposition 23. *Let $u\in\mathcal{L}(E)$ such as $|||u|||<1$. Then $\mathrm{Id}-u$ is inversible, and it's inverse is given by :*

$$u^{-1} = \sum_{n=0}^{+\infty} u^n \in \mathcal{L}(E).$$

Corollary 24. *The set $\mathrm{GL}(E)$ of continuous automorphisms of E is an open subset of $\mathcal{L}(E)$.*

2.3 L^p spaces.

Let $(X,\mathcal{A},\mu)$ be a measure space.

Definition 25. *For $0<p<\infty$ and f a complex valued measurable function defined on X, one can define :*

$$\|f\|_p := \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}.$$

The space $\mathcal{L}^p(\mu)$ is made up with the measurable functions $f: X\to\mathbb{C}$ such that $\|f\|_p<\infty$.

One can also define $\|.\|_\infty$ by $\|f\|_\infty := \inf\{a\in\mathbb{R}_+ : \mu(\{|f|>a\}=0)\}$, and $\mathcal{L}^\infty(\mu) =: \{f: X\to\mathbb{C} \text{ measurable}, \ \|f\|_\infty<\infty\}$.

Remark 26. If μ designates the counting measure on a countable space X, we usually denote $L^p(\mu)$ by ℓ^p.

Proposition 27. *(Hölder inequalty)*

For $p\in[1,\infty]$ and $q\in[1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}=\infty$, $f\in\mathcal{L}^p(\mu)$ and $g\in\mathcal{L}^q(\mu)$, one have :

$$\|fg\|_1 \leq \|f\|_p\cdot\|g\|_q$$

Corollary 28. *(Minkowski inequalty)*

For $f,g\in\mathcal{L}^p(\mu)$ with $p\in[1,\infty]$, one have $\|f+g\|_p\leq\|f\|_p+\|g\|_p$.

Corollary 29. *(Interpolation inequalty)*

Let $1\leq p,q\leq\infty$.

If $f\in\mathcal{L}^p(X,\mu)\cap\mathcal{L}^q(X,\mu)$ then $f\in L^r(X,\mu)$ for every $r\in[p,q]$ and :

$$\|f\|_{L^r} \leq \|f\|_p^\alpha\cdot\|f\|_q^{1-\alpha}$$

Definition 30. *Let $p\in[1,\infty]$. One can define the set $L^p(\mu)$ as the quotient set of $\mathcal{L}^p(\mu)$ by the equivalence relation given by $f\sim g \Longleftrightarrow \mu(\{f\neq g\})=0$.*

For convenience, we usually still denote f the class of f in $\mathcal{L}^p/\sim$.

Theorem 31. *For every $p\in[1,\infty]$, the space $(L^p,\|.\|_p)$ is a normed vector space.*

Développement 1 :

Theorem 32. *(Riesz-Fischer Theorem)*
For every $p \in [1, \infty]$, the space $(L^p, \|.\|_p)$ is complete. Therefore, L^p is a Banach space.

Proposition 33. *Let $p \in [1, \infty]$ and $(f_n)_{n \in \mathbb{N}}$ a Cauchy sequence in $L^p(\mu)$ converging to a limit $f \in L^p(\mu)$. There exist a sub-sequence $(f_{\varphi(n)})_{n \in \mathbb{N}}$ of $(f_n)_{\in \mathbb{N}}$ converging to f almost everywhere.*

2.4 Hilbert spaces.

Definition 34. *Let H be a vector space on the field $k = \mathbb{R}$ or $k = \mathbb{C}$. The space H is called an inner product space (or a pre-Hilbert space) if it's endowed with an inner product.*
If H is also a complete metric space, then H is called a Hilbert space.

Example 35. The set $\ell^2(\mathbb{N}) := \left\{ x = (x_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sum_{n=0}^{\infty} |x_n|^2 < \infty \right\}$ and more generally, the set $L^2(X, \mathcal{A}, \mu)$ are Hilbert spaces, with their respective inner products $\langle x, y \rangle_{\ell^2} = \sum_{n=0}^{+\infty} x_i \overline{y_i}$ and $\langle x, y \rangle_{L^2} = \int_X f \overline{g} \, d\mu$.

Theorem 36. *(Hilbert projection theorem)*
Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, and $C \subset H$ a nonempty closed convex subspace, and $x \in H$.
Then there exists a unique vector $y \in C$ for which $\|x - z\|$ is minimized over the vector $z \in C$. The vector y is noted $p_C(x)$ and called the projection of x on C.
Furthermore, the element $p_C(x)$ is characterized by the following :

1. *$p_C(x) \in C$,*
2. *$\forall z \in C \quad \mathfrak{Re}(\langle x - p_C(x), z - p_C(x) \rangle) \leq 0$.*

Corollary 37. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $F \subset H$ be a closed vector subset of H. Then we deduce from Theorem 36 that :*

$$H = \overline{F} \oplus (\overline{F})^{\perp} = \overline{F} \oplus F^{\perp}.$$

This formula induces a density criteria in a Hilbert space :

$$F \text{ is dense in } H \iff F^{\perp} = 0.$$

Theorem 38. *(Riesz representation theorem)*
Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. For every continuous linear map $\varphi \in H^*$, there exist a unique $f_\varphi \in H$ such that :

$$\forall x \in H \quad \varphi(x) = \langle f_\varphi, x \rangle = \langle x, f_\varphi \rangle.$$

Moreover, one have $\|f_\varphi\|_H = \|\varphi\|_{H^*}$.

Corollary 39. *(Radon-Nykodym theorem)*
Let μ and ν two σ-finite measures on $(X, \mathcal{A})$ such that $\mu(A) = 0 \Longrightarrow \nu(A) = 0$ for all $A \in \mathcal{A}$. Then there exists a measurable function $f : (X, \mathcal{A}) \to (\overline{\mathbb{R}}_+, \mathcal{B}(\overline{\mathbb{R}}_+))$ such that for every $A \in \mathcal{A}$:

$$\nu(A) = \int_X f \cdot \mathbf{1}_A \, d\mu.$$

3 Main applications of completeness.

3.1 Banach fixed point theorem and applications.

Definition 40. *Let (X, d) be a complete metric space. Then a map $T : X \to X$ is called a contraction mapping on X if there exists $q \in [0, 1]$ such that $d(T(x), T(y)) < q \cdot d(x, y)$.*

Theorem 41. *(Banach fixed point theorem)*
Let (X, d) be a nonempty complete metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed point $x^* \in X$. Furthermore, x^* can be found as follows : start with an arbitrary element $x_0 \in X$ and define the sequence $(x_n)_{n \in \mathbb{N}}$ by $x_{n+1} = T(x_n)$ for all $n \geq 0$. Then $x_n \underset{n \to +\infty}{\to} x^*$.

The Banach fixed point theorem has a crucial utility in the proof of some major theorems in calculus. We present some of them below.

Développement 2 :

Theorem 42. *(Cauchy-Lipschitz theorem)*
Let $U \subset \mathbb{R} \times \mathbb{R}^n$ an open subset, $\varphi : U \to \mathbb{R}$ continuous, locally lipschitz for it's second variable. Then for all $(t_0, u_0) \in U$, there exist an interval I such that $x_0 \in I$ and a function $f : I \to \mathbb{R}$ differentiable such that :

$$\begin{cases} \forall t \in I \quad f'(t) = \varphi(t, f(t)) \\ f(t_0) = u_0 \end{cases}.$$

Theorem 43. *(Lagrange multipliers theorem)*
Let U be an open subset of $\mathbb{R}^n$, $f, g_1, \ldots, g_k : U \to \mathbb{R}^n$ be functions of class $\mathcal{C}^1$, and $M := \{x \in U : g_1(x) = \cdots = g_k(x) = 0\}$. Let suppose that $f_{|M}$ has a local extremum in an element $m \in M$ and that the differential forms $(Dg_i(m))_{0 \leq i \leq k}$ are lineary independant.
Then there exists $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that :

$$Df(m) = \sum_{i=0}^{k} \lambda_i Dg_i(m).$$

The numbers $(\lambda_1, \ldots, \lambda_k)$ *are called the Lagrange multipliers associated to* $(Dg_i(m))_{0 \leq i \leq k}$ *and* f.

Theorem 44. *(Implicit function theorem)*
Let U *be an open subset of* $\mathbb{R}^n \times \mathbb{R}^p$, (a,b) *an element of* U, *and* $f\colon \begin{cases} U & \to \mathbb{R}^p \\ (x,y) & \mapsto f(x,y) \end{cases}$ *a function of class* $\mathcal{C}^1$. *Let's suppose that* $f(a,b)=0$ *and that the jacobian matrix* $D_y f(a,b)$ *(made up with the partial derivatives with respect to* y*) is inversible, i.e* $\det D_y(a,b) \neq 0$.
Then the equation $f(x,y)=0$ *can be locally resolved with respect to the variables* y *: there exists* V *(open neighborhood of* a *in* $\mathbb{R}^n$*) and* W *(open neighborhood of* b *in* $\mathbb{R}^p$*) with* $V \times W \subset U$, *and a mapping* $\varphi\colon V \to W$ *of class* $\mathcal{C}^1$, *such that :*

$$(x \in V,\ y \in W \text{ and } f(x,y)=0) \Longleftrightarrow (x \in V \text{ and } y = \varphi(x)).$$

Furthermore, $D_y f(x,y)$ *is inversible for all* $(x,y) \in V \times W$.

3.2 Baire theorem and applications.

Theorem 45. *Let* (E,d) *be a complete metric space. For a sequence* $(\mathcal{O}_n)_{n\in\mathbb{N}}$ *of open subsets dense in* E, *the subset* $\bigcap_{n\in\mathbb{N}} \mathcal{O}_n$ *is still dense in* E.

Remark 46. An equivalent statement is the following : for a sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of closed subset of E with empty interior, the subset $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ still has empty interior.

Corollary 47.
A vector space with a countable basis is not a complete metric space.

Corollary 48. *Let* $f\colon \mathbb{C} \to \mathbb{C}$ *an holomorphic mapping such that :*

$$\forall z \in \mathbb{C}, \quad \exists n \in \mathbb{N}, \quad f^{(n)}(z) = 0,$$

then f *is a polynomial function.*

The Baire theorem also has many applications in functionnal analysis, such as the following theorems for Banach spaces.

Theorem 49. *(Open mapping theorem)*
Let E, F *be two Banach spaces and* $T\colon E \to F$ *be a continuous linear surjective mapping. There exists* $M \in (0,\infty)$ *such that* $T(B_E(0,1)) \supset B_F(0, M^{-1})$, *i.e :*

$$\forall y \in F \quad \exists x \in E \text{ with } y = T(x) \text{ and } \|x\| \leq M \cdot \|y\|.$$

Theorem 50. *(Banach-Steinhaus theorem or uniform boundedness principle)*
Let E, F *be two vector spaces on a field* K, *with* E *being a complete metric space. Let* $(T_i)_{i\in I}$ *be continuous linear mapping of* $\mathcal{L}(E,F)$. *Let's suppose that* $\forall x \in E \quad \sup_{i\in I} \|T_i(x)\| < \infty$.

Then $\sup_{i\in I} \|T_i\| < \infty$.

Corollary 51. *The set of continuous* 2π*-periodic functions whose Fourier series in zero is dense is* $\mathcal{C}^0_{2\pi}$.

3.3 Continuous extension theorems.

Theorem 52.
Let A *be a dense subset of a metric space* (X,d) *and a uniformly continuous mapping from* A *to a metric space* (Y,d'). *Then :*

1. *There exists a unique continuous extension of* f *to* X.
2. *This extension is uniformly continuous.*

Corollary 53.
Let F *be a vector space, and* G *be a dense subset of* F. *Then every continuous linear mapping* $f\colon G \to E$ *has a unique continuous linear extension* $\tilde{f}\colon E \to E$.

Example 54. *(Construction of Rieman integral)*
Let $a \leq b$ be reals numbers. We denote $\mathcal{E}(a,b)$ the set of simple functions on $[a,b]$. We define the Riemann integral of $f \in \mathcal{E}(a,b)$ by :

$$\mathcal{I}_{[a,b]}(f) := \int_a^b f(x)\,dx := \sum_{k=0}^{n-1} (x_{i+1} - x_i)\,c_i.$$

Where $\sigma = (x_i)_{0 \leq i \leq n}$ is a partition of the interval $[a,b]$ adapted to f and c_i is the value of f on the interval (x_i, x_{i+1}). Then the mapping $f \mapsto \mathcal{I}_{[a,b]}(f)$ is uniformly continuous from $\mathcal{E}(a,b)$ to $\mathbb{R}$, and $\mathcal{E}(a,b)$ is dense in $\mathcal{C}^0([a,b])$. This allow us to extend the Riemann integral to continuous functions on $[a,b]$.

Example 55. *(Construction of conditionnal expectation)*
Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra of $\mathcal{F}$ and $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ an integrable random variable. We call conditionnal expectation of X given $\mathcal{G}$ every random variable $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that :

1. Y is $\mathcal{G}$-measurable,
2. $\forall A \in \mathcal{G} \quad \mathbb{E}[Y \cdot \mathbf{1}_A] = \mathbb{E}[X \cdot \mathbf{1}_A]$.

Then, the conditionnal expectation of $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ given $\mathcal{G}$ exists, and is unique. It's denoted $\mathbb{E}[X|\mathcal{G}]$.

One method to prove this theorem is by using the good properties of Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$, and then extend the conditionnal expectation to $L^1(\Omega, \mathcal{F}, \mathbb{P})$ by continuity, using Corollary 53.

Example 56. *(Extension of Fourier transform to $L^2(\mathbb{R})$)*

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space, made up with function whose derivatives are rapidly decreasing.

We define the Fourier transform of $f \in \mathcal{S}(\mathbb{R})$ by $\mathcal{F}(f)(\xi) := \int_{\mathbb{R}} f(x)\, e^{-2i\pi x\xi}\, dx$. Thus holds the Plancherel theorem :

$$\int_{\mathbb{R}} \mathcal{F}(f)(\xi)\, \overline{\mathcal{F}(g)(\xi)}\, d\xi = \int_{\mathbb{R}} f(x)\bar{g}(x)\, dx \text{ and} \int_{\mathbb{R}} |\mathcal{F}(f)(\xi)|^2\, d\xi = \int_{\mathbb{R}} |f(x)|^2\, dx.$$

One can prove that $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$: therefore, with the continuous extension theorem and the Plancherel theorem, one can define a continuous extension of Fourier transform $\mathcal{F}$ on $L^2(\mathbb{R})$.